A Terracini Lemma for osculating spaces with applications to Veronese surfaces

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Abstract

Here we present a partial generalization to higher order osculating spaces of the classical Lemma of Terracini on ordinary tangent spaces. As an application, we investigate the secant varieties to the osculating varieties to the Veronese embeddings of the projective plane.

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1 Introduction

The geometry of defective varieties, whose secant varieties have dimension less than the expected one, is a subtle and intriguing subject which has been investigated by several authors, both classical and modern (see for instance the introduction and the references in [5]). The main tool for understanding defective varieties is provided by the celebrated Lemma of Terracini (see [12] for the original statement and [10], [1] for modern versions):

Lemma 1. *Let* $X \subset \mathbb{P}^r$ *be an integral non-degenerate projective variety of dimension* n *and let* $h \leq r$ *be a positive integer. Take* $h+1$ *general points* $p_0, \ldots, p_h$ *of* X *and let* $P \in < p_0, \ldots, p_h >$ *be a general point in the* h*-secant variety* $\mathrm{Sec}_h(X)$ *of* X*. Then the tangent space* $T_P(\mathrm{Sec}_h(X))$ *to* $\mathrm{Sec}_h(X)$ *at* P *is given by* $T_P(\mathrm{Sec}_h(X)) =< \bigcup_{i=0}^{h} T_{p_i}(X) >$.

The generalization of the above result from tangent spaces to higher order osculating spaces (respectively, from double points to points of higher multeplicity) is a very delicate problem, which has recently attracted a great deal of interest (for instance, we are aware of work in progress by Ciliberto on these topics). Here we present our attempts in this direction, which have

been inspired by the less known paper [13] of Terracini: indeed, our Lemma 2 should be regarded as a rough extension of Lemma 1.

As an application, we are going to investigate the secant varieties to the osculating varieties $T(m, V_{2,d})$ of order m to the d-Veronese surface $V_{2,d}$. It is known that the tangential variety to $V_{2,d}$ is not h-defective unless $d = 3$ and $h = 1$ (see [7] and [2]) and that the 2-osculating variety of $V_{2,d}$ is not h-defective unless $d = 4$ and $h = 1$ (see [3]). Other related results are collected in the very recent pre-print [6]. Here instead we prove the following:

Proposition 1. *Fix integers $d \geq 1$, $h \geq 1$, $m \geq 1$. If*

$$\frac{(d+2)(d+1)}{2} \leq 2(h+1) + (h+1)\frac{(m+2)(m+1)}{2}$$

then $T(m, V_{2,d})$ is h-defective for:
(a) $h = 1$, $m + 2 \leq d \leq 2m + 2$;
(b) $h = 2$, $3(m + 2)/2 \leq d \leq 2m + 2$;
(c) $h = 4$, $2m + 4 \leq d \leq (5m + 8)/2$;
(d) $h = 5$, $12(m + 2)/5 \leq d \leq (5m + 8)/2$;
(e) $h = 6$, $21(m + 2)/8 \leq d \leq (8m + 14)/3$;
(f) $h = 7$, $48(m + 2)/17 \leq d \leq (17m + 32)/6$.
If instead

$$\frac{(d+2)(d+1)}{2} \geq 2(h+1) + (h+1)\frac{(m+2)(m+1)}{2}$$

then $T(m, V_{2,d})$ is h-defective for:
(a) $h = 1$, $m + 1 \leq d \leq 2m$;
(b) $h = 2$, $3(m + 1)/2 \leq d \leq 2m$;
(c) $h = 4$, $2m + 2 \leq d \leq (5m + 3)/2$;
(d) $h = 5$, $12(m + 1)/5 \leq d \leq (5m + 3)/2$;
(e) $h = 6$, $21(m + 1)/8 \leq d \leq (8m + 6)/3$;
(f) $h = 7$, $48(m + 1)/17 \leq d \leq (17m + 15)/6$.

Theorem 1. *Fix integers $d \geq 1$, $h \geq 1$, $m \leq 18$. Then*

$$\dim(\mathrm{Sec}_h(T(m, V_{2,d}))) = \mathrm{expdim}(\mathrm{Sec}_h(T(m, V_{2,d}))) =$$

$$= \min\left\{2(h+1) + (h+1)\frac{(m+2)(m+1)}{2} - 1, \frac{(d+2)(d+1)}{2} - 1\right\}$$

for
(a) $h = 1$ and either $d < m + 1$ or $d > 2m + 2$;
(b) $h = 2$ and either $d < 3(m + 1)/2$ or $d > 2m + 2$;

(c) $h = 4$ *and either* $d < 2m + 2$ *or* $d > (5m + 8)/2$*;*
(d) $h = 5$ *and either* $d < 12(m + 1)/5$ *or* $d > (5m + 8)/2$*;*
(e) $h = 6$ *and either* $d < 21(m + 1)/8$ *or* $d > (8m + 14)/3$*;*
(f) $h = 7$ *and either* $d < 48(m + 1)/17$ *or* $d > (17m + 32)/6$*;*
(g) $h = 3$ *or* $h \geq 8$*.*

We deduce the above statements from the classification of (-1)-special linear systems on the projective plane due to Ciliberto and Miranda ([9]). Indeed, our proof shows that Theorem 1 holds for every integer m such that any special linear system of plane curves with base points of equal multiplicity $m + 2$ is (-1)-special.

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2 The results

Let $X \subset \mathbb{P}^r$ be a non-degenerate integral projective variety of dimension n defined over the complex field $\mathbb{C}$. If $p \in X$ and m is a non-negative integer, let $T_p^m(X)$ denote the m-osculating space to X at p (in particular, we have $T_p^0(X) = \{p\}$ and $T_p^1(X) = T_p(X)$, the usual tangent space). Fixed non-negative integers $m_0, \ldots, m_h$, define the higher order join

$$J(m_0, \ldots, m_h, X) := \overline{\bigcup_{p_0, \ldots, p_h} < T_{p_0}^{m_0}(X), \ldots, T_{p_h}^{m_h}(X) >},$$

where $p_0, \ldots, p_h$ are general points on X. In particular, if $m_0 = \ldots = m_h = m$, we have

$$J(m, \ldots, m, X) = \mathrm{Sec}_h(T(m, X)),$$

where $T(m, X)$ is the m-osculating variety of X (see for instance [4], Definition 1) and Sec_h denotes the h-secant variety (see for instance [11], Definition 1.1).

Lemma 2. *Notation as above.*

(i) The expected dimension of $J(m_0, \ldots, m_h, X)$ *is*

$$\mathrm{expdim}(J(m_0, \ldots, m_h, X)) = \min \left\{ (h + 1)n + \sum_{i=0}^{h} \binom{m_i + n}{n} - 1, r \right\}.$$

(ii) There is a natural inclusion

$$T_P(J(m_0,\dots,m_h,X)) \subseteq < T^{m_0+1}_{p_0}(X),\dots,T^{m_h+1}_{p_h}(X) >,$$

where $p_0,\dots,p_h$ are general points on X and P is general in $< T^{m_0}_{p_0}(X),\dots,$ $T^{m_h}_{p_h}(X) >$.

(iii) If $r \geq (h+1)n+\sum_{i=0}^h \binom{m_i+n}{n}-1$ and $\dim < T^{m_0}_{p_0}(X),\dots,T^{m_h}_{p_h}(X) >= \sum_{i=0}^h \binom{m_i+n}{n}-1-\delta$, then

$$\dim J(m_0,\dots,m_h,X)) \leq (h+1)n+\sum_{i=0}^h \binom{m_i+n}{n}-1-\delta.$$

(iv) If $r \geq \sum_{i=0}^h \binom{m_i+n+1}{n}-1$ and $\dim < T^{m_0+1}_{p_0}(X),\dots,T^{m_h+1}_{p_h}(X) >= \sum_{i=0}^h \binom{m_i+n+1}{n}-1$, then

$$\dim(J(m_0,\dots,m_h,X)) = \operatorname{expdim}(J(m_0,\dots,m_h,X)).$$

(v) If $n=2$, $\dim < T^{m}_{p_0}(X),\dots,T^{m}_{p_h}(X) >= \min\left\{(h+1)\frac{(m+2)(m+1)}{2}-1, r\right\}$ and $\dim \mathrm{Sec}_h(T(m,X)) < \min\left\{2(h+1)+(h+1)\frac{(m+2)(m+1)}{2}-1, r\right\}$, then

$$T_P(\mathrm{Sec}_h(T(m,X))) =< T^{m+1}_{p_0}(X),\dots,T^{m+1}_{p_h}(X) >,$$

where $p_0,\dots,p_h$ are general points on X and P is general in $< T^{m}_{p_0}(X),\dots,$ $T^{m}_{p_h}(X) >$.

Proof. For $0 \leq i \leq h$ let

$$\begin{array}{rcl} p_i : U_i \subseteq \mathbb{C}^n & \longrightarrow & X \\ t_{i1},\dots,t_{in} & \longmapsto & p_i(t_{i1},\dots,t_{in}) \end{array}$$

be a local parametrization of X (in the euclidean topology) centered in a general point of X. A general point of $J(m_0,\dots,m_h,X)$ is of the form:

$$P = p_0(t_{01},\dots,t_{0n}) + \sum_{i=1}^h \lambda_i p_i(t_{i1},\dots,t_{in}) + \sum_{\substack{i=0 \\ 1\leq |I| \leq m_i}}^h \lambda_i^I p_i^I(t_{i1},\dots,t_{in})$$

where p^I denotes the derivative of p corresponding to the multi-index $I=(a_1,a_2,\dots)$ with every $a_j \in \{1,\dots,n\}$. Notice that P depends on $(h+1)n+\sum_{i=0}^h \binom{m_i+n}{n}-1$ parameters, hence (i) follows.

The tangent space $T_P(J(m_0, \ldots, m_h))$ is the linear span

$$< \{P\} \cup \left\{ \frac{\partial P}{\partial t_{ij}} \right\}_{\substack{i=0,\ldots,h \\ j=1,\ldots,n}} \cup \left\{ \frac{\partial P}{\partial \lambda_i} \right\}_{i=0,\ldots,h} \cup \left\{ \frac{\partial P}{\partial \lambda_i^I} \right\}_{\substack{i=0,\ldots,h \\ 1 \leq |I| \leq m_i}} >=$$

$$=< \{p_i\}_{i=0,\ldots,h} \cup \{p_i^I\}_{\substack{i=0,\ldots,h \\ 1 \leq |I| \leq m_i}} \cup \left\{ \sum_{|I|=m_i} \lambda_i^I p_i^{I \cup \{j\}} \right\}_{\substack{i=0,\ldots,h \\ j=1,\ldots,n}} >,$$

hence (ii), (iii), and (iv) follow.

Finally we have to check (v). Since $< T_{p_0}^m(X), \ldots, T_{p_h}^m(X) >$ is of the expected dimension but $\mathrm{Sec}_h(T(m, X))$ is not, there is at least one $i \in \{0, \ldots, h\}$ and $j(i) \in \{1, 2\}$ such that $\sum_I \lambda_i^I p_i^{I \cup \{j(i)\}}$ is a linear combination of the other points spanning $T_P(\mathrm{Sec}_h(T(m, X))$. Since the coefficients λ_i^I are general, by specializing all but one coefficients λ_i^I to 0 and the remaining one to 1 in all possible ways, we obtain:

$$\begin{aligned} T_P(\mathrm{Sec}_h(T(m, X)) &= < \{p_k\}_{k=0,\ldots,h} \cup \{p_k^I\}_{\substack{k=0,\ldots,h \\ 1 \leq |I| \leq m}} \cup \\ &\quad \cup \{p_i^I\}_{|I|=m+1} \cup \left\{ \sum_{|I|=m} \lambda_k^I p_k^{I \cup \{j\}} \right\}_{\substack{k \neq i \\ j=1,\ldots,n}} > . \end{aligned}$$

Moreover, since $m_0 = \ldots = m_h = m$, the same is true for every $i \in \{0, \ldots, h\}$, hence the claim follows.

$\square$

Now we turn to the promised applications. We point out that our arguments rely on the main classification result of [9], from which we borrow also notation and terminology.

Proof of Proposition 1. In the former case, we have

$$\mathrm{expdim}(\mathrm{Sec}_h(T(m, V_{2,d}))) = \frac{(d+1)(d+2)}{2} - 1$$

for every $m \geq 1$, hence the claim follows directly from Lemma 2 (ii) and the speciality of the linear system $\mathcal{L}_d((m+2)^{h+1})$ (see [8], Theorem 2.4). In the latter case, the claim follows from Lemma 2 (iii) and the speciality of the linear system $\mathcal{L}_d((m+1)^{h+1})$ (see again [8], Theorem 2.4).

$\square$

Proof of Theorem 1. Notice first of all that, since $m+2 \leq 20$, any special $\mathcal{L}_d((m+2)^{h+1})$ is (-1)-special by [8]. Next, if $\frac{(d+1)(d+2)}{2} \geq (h+1)\frac{(m+3)(m+2)}{2}$, just apply Lemma 2 (iv). Finally, if $\frac{(d+1)(d+2)}{2} \leq (h+1)\frac{(m+3)(m+2)}{2}$, argue by contradiction and conclude by Lemma 2 (v).

$\square$

References

[1] B. Ådlandsvik: Joins and higher secant varieties, Math. Scand. 62 (1987), 213–222.

[2] E. Ballico, On the secant varieties to the tangent developable of a Veronese variety. Pre-print (2003).

[3] E. Ballico and C. Fontanari: On the secant varieties to the osculating variety of a Veronese surface. Cent. Eur. J. Math. 1 (2003), 315–326.

[4] E. Ballico and C. Fontanari: On a Lemma of Bompiani. Rend. Sem. Mat. Univ. Politec. Torino (to appear).

[5] E. Ballico and C. Fontanari: Birational geometry of defective varieties. Pre-print (2003).

[6] A. Bernardi, M. V. Catalisano, A. Gimigliano, M Idà: Osculating varieties of Veronesean and their higher secant varieties. Pre-print math.AG/04033132 (2004).

[7] M. V. Catalisano, A. V. Geramita, A. Gimigliano: On the secant variety to the tangential varieties of a Veronesean, Proc. Amer. Math. Soc. 130 (2001), no. 4, 975–985.

[8] C. Ciliberto, F. Cioffi, R. Miranda, F. Orecchia: Bivariate Hermite Interpolation and Linear Systems of Plane Curves with Base Fat Points, Proceedings ASCM 2003, Lecture Notes Series on Computing 10, World Scientific Publ., Singapore/River Edge, USA (2003), 87–102.

[9] C. Ciliberto and R. Miranda: Linear systems of plane curves with base points of equal multiplicity. Trans. Amer. Math. Soc. 352 (2000), no. 9, 4037–4050.

[10] M. Dale: Terracini's lemma and the secant variety of a curve. Proc. London Math. Soc. (3), 49 (1984), 329-339.

[11] C. Dionisi and C. Fontanari: Grassmann defectivity à la Terracini. Matematiche (Catania) 56 (2001), 245–255.

[12] A. Terracini: Sulle V_k per cui la varietà degli S_h-$h+1$ seganti ha dimensione minore dell'ordinario. Rend. Circ. Mat. Palermo 31 (1911), 392-396.

[13] A. Terracini: Sulle superficie i cui spazi osculatori presentano particolari incidenze coi piani tangenti o fra loro. Atti Soc. Natur. e Matem. Modena, V, 6 (1922), 34-58.

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